FURTHER PROPERTIES OF ABELIAN INTEGRALS ATTACHED TO ALGEBRAIC VARIETIES

By W. V. D. HODGE

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated November 12, 1931

In [illegible] note[1] I made use of the idea of the product of two constructs in order to obtain certain properties of Abelian integrals attached to algebraic varieties. The present note uses the same method to obtain further properties of such integrals.

§1.

1. Let A, B be two homeomorphic absolute manifolds of n dimensions, and consider the product $A \times B$. On this there is a cycle Γ of n dimensions, homeomorphic to A or B, corresponding to the transformation between A and B implied by their homeomorphism. If a_p^i $(i = 1, \ldots, R_p)$, where R_p is the pth Betti number of A, is a base for the p-cycles of A, and b_p^i $(i = 1, \ldots, R_p)$ is the *corresponding* base for the p-cycles of B, then

$$a_p^i \times b_{r-p}^j \ (i = 1, \ldots, R_p;\ i = 1, \ldots, R_{r-p};\ p = 0, \ldots r)$$

form a base for the r-cycles of $A \times B$. We therefore have

$$\Gamma \approx \sum_{i,j,p} \epsilon_{ij}^p \, a_p^i \times b_{n-p}^j.$$

The coefficients ϵ_{ij}^p are known,[2] and the matrix $(\epsilon_{ij}^p) = \epsilon^p$ is given by $\epsilon^p = (\overline{a^p})^{-1}$ where $\overline{a_{ji}^p} = a_{ij}^p = a_p^i . a_{n-p}^j$ (the intersection of a_p^i and a_{n-p}^j). Now let Γ_r^h be the r-cycle of Γ which is the homeomorph of the cycle a_r^h of A.

$$\begin{aligned} \Gamma_r^h &\approx \Gamma . a_r^h \times B \\ &\approx \sum_{i,j,p} \epsilon_{ij}^p \, a_p^i . a_r^h \times b_{n-p}^j \approx \sum_{i,j,p} (-1)^{pr} \epsilon_{ij}^p a_r^h . a_p^i \times b_{n-p}^j. \end{aligned}$$

Let $$a_r^h . a_p^i \approx \sum_l \mu_{hil}^{rp} \, a^l{}_{(r+p-n)} \quad (\text{or zero if } r + p < n).$$

Then $$a_r^h . a_p^i . a_{2n-r-p}^j = \sum \mu_{hil}^{rp} \, a^l{}_{(r+p-n)} . a_{2n-r-p}^j.$$

The matrices $$a_h^{rp} = (a_r^h . a_p^i . a_{2n-r-p}^j),$$

and $$a^{r+p-n} = (a_{r+p-n}^i . a_{2n-r-p}^j)$$

are characteristic matrices of the manifold and are known when the manifold is completely specified. Hence if

$$\mu_h^{rp} = (\mu_{hij}^{rp}),$$

$$\mu_h^{rp} = a_h^{rp}(a^{r+p-n})^{-1}.$$

Now
$$\Gamma_r^h \approx \sum (-1)^{pr} \epsilon_{ij}^p \mu_{hil}^{rp} a_{r+p-n}^l \times b_{n-p}^j$$
$$\approx \sum \lambda_{lj}^{r+p-n} a_{r+p-n}^l \times b_{n-p}^j,$$

where
$$\lambda^{r+p-n} = (-1)^{pr} \overline{(\mu_h^{rp})} \epsilon^p$$
$$= (-1)^{pr} \overline{(a^{r+p-n})}^{-1} \overline{(a_h^{rp})} \overline{(a^p)}^{-1},$$

and hence Γ_r^h can be expressed in terms of the base for r-cycles of $A \times B$ by a homology in which the coefficients are known.

A more convenient method of writing the result is to put

$$p + q = r.$$

Then
$$\lambda^p = (-1)^{(n-q)r} \overline{(a^p)}^{-1} \overline{(a_h^{r,n-q})} \overline{(a^{n-q})}^{-1}.$$

2. Let the manifolds A and B be two Riemann manifolds associated with the algebraic variety F whose equation is

$$F(x_0, x_1, \ldots, x_m) = 0$$

in a (complex) space of $m + 1$ dimensions in which the coördinates are $(x_0, \ldots, x_m)$. $A \times B$ is then a Riemann manifold associated with the ordered product Φ of the variety by itself, or, if we like, the intersection of the cylinders

$$F(x_0, x_1, \ldots, x_m) = 0$$
$$F(x_0', x_1', \ldots, x_m') = 0$$

in the space of $2m + 2$ dimensions $(x_0, \ldots, x_m, x_0', \ldots, x_m')$.

If
$$\int_{(p)} \sum_{i_q=1}^{m} P_{i_1 i_2 \ldots i_p} dx_{i_1} \ldots dx_{i_p} = \int_{(p)} dP$$

is a p-fold integral of a total differential of the first kind attached to F, and

$$\int_{(q)} \sum_{j_r=1}^{m} Q_{j_1 \ldots j_q}\, dx_{j_1} \ldots dx_{j_q} = \int_{(q)} dQ$$

is a q-fold integral of similar type,

$$\int_{(p+q)} \sum_{i,j} P_{i_1 \ldots i_p} Q'_{j_1 \ldots j_q}\, dx_{i_1} \ldots dx_{i_p}\, dx'_{j_1} \ldots dx'_{j_q}, \qquad (1)$$

where the "Q'" implies that each x_i is replaced by x_i' in Q, is a $(p + q)$-fold integral of a total differential on Φ. The variety of m dimensions on Φ which corresponds to the identity transformation on F is given by putting $x_i = x_i'(i = 0, \ldots, m)$ and is represented on $A \times B$ by Γ. The value of the integral (1) taken over a cycle Γ_r^h $(r = p + q)$ of Γ is equal to the value of

$$\int_{(p+q)} \sum_{i,j} P_{i_1} \ldots {}_{i_p} Q'_{j_1} \ldots {}_{j_q} dx_{i_1} \ldots dx_{i_p} dx_{j_1} \ldots dx_{j_q}$$

taken over the corresponding cycle of A, and hence it vanishes if $r > m$. But as (1) is the integral of a total differential of the first kind on $A \times B$ its value is unaltered if Γ_r^h is replaced by a homologous cycle, in particular by the cycle

$$\sum_{i,j,p} \lambda_{ij}^p a_p^i \times b_{r-p}^j.$$

Now the value of the integral evaluated over

$$a_p^i \times b_{r-p}^j$$

is $\omega_i^p \times \omega_j^q$, where

$$\omega_i^p = \int_{a_p^i} dP, \text{ and } \omega_j^q = \int_{b_q^j} dQ',$$

and the value over $a_{p'}^i \times b_{r-p'}^j$ $(p \neq p')$ is zero, since this cycle can be deformed into one lying in

$$x_{p'+1} = 0, \ldots, x_m' = 0, x_{r-p'+1}' = 0, \ldots, x_m' = 0,$$

and, as either $p' < p$ or $r - p' < r - p$, the result follows immediately.

Hence we have

$$0 = \sum_{i,j} \lambda_{ij}^p \omega_i^p \omega_j^q.$$

If ω^p denotes the matrix whose element in the i^{th} column and the j^{th} row is the period of the i^{th} p-fold integral of a total differential on A with respect to the cycle a_p^j, we have the matrix equation

$$\omega^p \lambda^p \overline{\omega^q} = 0, \tag{2}$$

and there will be one such relation for each of the R_r r-cycles of Γ.

3. The only question which remains is whether the relations (2) are vacuous or not. Consider the simple case in which $m = 2$, so that F is an algebraic surface, and take $p = 1$, $q = 2$. The relation (2) is made up of relations of the form

$$\sum_{i,j} \lambda_{ij}^1 \omega_i^1 \omega_j^2 = 0,$$

where ω_i^1 $(i = 1, \ldots, R_1)$ and ω_j^2 $(j = 1, \ldots, R_2)$ are the periods of a

Picard integral of the first kind and of a double integral of the first kind, respectively, attached to F. We know that $\omega_i^2 = 0$ if a_2^i is an algebraic cycle, and what we require to show is that there exist cycles a_1^i, a_2^j such that $\omega_i^1 \times \omega_j^2 \neq 0$, and further $\lambda_{ij}^1 \neq 0$. A simple example will show that this is the case, and hence the relation (2) does in fact yield relations between the periods of the p- and q-fold integrals of total differentials of the first kind on F.

Let C and D be two elliptic curves, and denote their 0-cycles; 1-cycles; and 2-cycles, by c; γ_1, γ_2; C and d; δ_1, δ_2; D, respectively. Then take F as the product of $C \times D$. On F the 0-cycle is $a_0 = c \times d$, the 1-cycles $a_1^1, \ldots, a_1^4$ are $c \times \delta_1$, $c \times \delta_2$, $\gamma_1 \times d$, $\gamma_2 \times d$, the two cycles $a_2^1, \ldots, a_2^6$ are $c \times D$, $\gamma_1 \times \delta_1$, $\gamma_1 \times \delta_2$, $\gamma_2 \times \delta_1$, $\gamma_2 \times \delta_2$, $C \times d$, and the 3-cycles $a_3^1, \ldots, a_3^4$ are $\gamma_1 \times D$, $\gamma_2 \times D$, $C \times \delta_1$, $C \times \delta_2$. The matrices

$$a^1 = \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix}, \qquad a^2 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$(a_2^i \times a_3^j) = \begin{bmatrix} 0 & 0 & c \times \delta_1 & c \times \delta_2 \\ 0 & c \times \delta_1 & 0 & -\gamma_1 \times d \\ 0 & c \times \delta_2 & \gamma_1 \times d & 0 \\ -c \times \delta_1 & 0 & 0 & -\gamma_2 \times d \\ -c \times \delta_2 & 0 & \gamma_2 \times d & 0 \\ \gamma_1 \times d & \gamma_2 \times d & 0 & 0 \end{bmatrix}.$$

Now let Γ_r^h be the homeomorph of a_3^1, then a_h^{r2} is

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix},$$

and from this it follows that

$$\lambda^1 = \begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

If $(1, \tau)$ are the periods of the integral of the first kind attached to C and

$(1, \sigma)$ those of the integral on D, the period matrix for Picard integrals of the first kind on F is

$$\begin{matrix} 1 & \sigma & 0 & 0 \\ 0 & 0 & 1 & \tau \end{matrix}$$

and the periods of the double integral of the first kind are

$$0,\ 1,\ \sigma,\ \tau,\ \tau\sigma,\ 0;$$

and hence if we take the relation arising from the first Picard integral

$$\lambda_{13}^{1} = -1,\ \omega_{1}^{1} = 1,\ \omega_{3}^{2} = \sigma,$$

and this establishes the required result.

4. If there is a singular transformation of F into itself we shall get a new algebraic cycle Γ on $A \times B$ corresponding to this, and we shall have new relations of type (2) arising from each Γ_r^h of this Γ. Lefschetz[3] has shown how to calculate the coefficients $\epsilon_{ij}'^{p}$ for Γ in this case. Making the necessary alterations in our calculations, if

$$\Gamma_r^h \approx \sum_{i,j,p} \mu_{ij}^{p}\, a_p^i \times b_{r-p}^{j},$$

$$\mu^p = \lambda^p f^q \quad (r = p + q)$$

when f^q is the matrix of transformation of the q-cycles. The new relation is then

$$\omega^p \lambda^p f^q \overline{\omega^q} = 0.$$

But the Hurwitz[4] theory of correspondence tells us that there is a matric ν^q such that

$$\nu^q \omega^q = \omega^q \overline{f^q},$$

and hence the new relations can be obtained by combining the relations (2) with the Hurwitz relations.

§2.

5. The second application of our method deals with integrals of the second kind attached to an algebraic variety. It is known that if

$$\int \phi_i(x, y)dx \quad (i = 1, \ldots, 2p)$$

are $2p$ independent integrals of the second kind attached to a curve C of genus p

$$f(x, y) = 0,$$

and (x_1, y_1) and (x_2, y_2) are two points of C, then there exists a certain matrix (ϵ_{ij}), whose elements do not depend on (x_1, y_1) or (x_2, y_2) such that

$$\sum_{ij} \epsilon_{ij}\, \phi_i(x_1, y_1)\, \phi_i\, (x_2, y_2) = \frac{\partial}{\partial x_1} R(x_1, x_2, y_1, y_2) + \frac{\partial}{\partial x_2} S(x_1, x_2, y_1, y_2)$$

where R and S denote rational functions; and the differentiations are made on the assumption that (x_1, y_1) and (x_2, y_2) lie on C. The importance of this result lies in the fact that it gives an algebraic form to the well-known theory of the "interchange of argument and parameter" for normal integrals of the third kind.[5]

The existence of such a result is obvious from certain geometrical considerations, but not its form. The argument is as follows. We take C as the variety F of §1 and consider the surface Φ. Denote the integrals of the second kind attached to A by $U_1, U_2, \ldots, U_{2p}$, and let the corresponding ones attached to B be $V_1, V_2, \ldots, V_{2p}$. Then clearly $U_i \times V_j$ is an integral of the second kind attached to Φ. Now for Φ

$$R_2 = 4p^2 + 2.$$

Consider the Picard number ρ. This is in general equal to 3, for there are three independent curves on it $x =$ const., $x' =$ const. $x = x'$. Therefore there are

$$R_2 - \rho = 4p^2 - 1$$

independent integrals of the second kind on Φ, and hence the $4p^2$ integrals $U_i \times V_j$ are not all independent. There must therefore be a matrix (ϵ_{ij}) such that

$$\sum_{ij} \epsilon_{ij} U_i \times V_j$$

is improper.

6. This suggests generalizations of the result. In the first place, if there are k singular correspondences on the curve C, to each of these there corresponds a curve on Φ (or rather two curves on Φ, which are interchanged when we interchange x and x'). Hence ρ is increased, so that to each correspondence there corresponds a new bilinear expression in U, V which is improper. Again, the arguments can be extended to any variety and give similar results for varieties of m dimensions. The application of the results to function theory is, however, not obvious.

By means of our topological methods we are able to give exact forms to the bilinear expressions which are improper, and this paragraph is devoted to finding these forms. We proceed with the most general case and use the notation of §1.

7. Consider on $A \times B$ the algebraic cycle of $2m$ dimensions Γ which represents any correspondence between A and B,

$$\Gamma \approx \sum_{i,j,p} \epsilon_{ij}^p a_p^i \times b_{2m-p}^j.$$

From the formula of Lefschetz, already quoted,

$$\epsilon^p = (\bar{a}^p)^{-1} f^{2m-p}$$

where f^{2m-p} is the matrix of transformation of the $(2m - p)$-cycles. Let Γ_1 stand for the cycle

$$\Gamma_1 \approx \sum_{ij} \epsilon_{ij}^m a_m^i \times b^j m.$$

Then $(\Gamma_1 . \Gamma_1) = \gamma = (-1)^m \sum_{i,j,h,k} \epsilon_{ij} \epsilon_{hk} a_{ih} b_{jk}$ (where the affix m is omitted, as this is the only one which is required in the sequel).

Writing
$$\Gamma_{rs} = \Gamma_1 \sum_{ij} \epsilon_{ij} a_{ir} b_{js} - (-1)^m \gamma a_m^r \times b_m^s,$$
we have
$$(\Gamma_{rs} . \Gamma_1) = 0, \ \sum \epsilon_{rs} \Gamma_{rs} \approx 0.$$

Let $U_1, \ldots, U_{rm}$ be the integrals of the second kind attached to A which have unit periods on $a_m^1, \ldots, a_m^{Rm}$, respectively, and all their other periods equal to zero, and let $V_1, \ldots, V_{Rm}$ be the corresponding integrals of B. Some of these may be improper, but this does not affect the argument. The period of $U_a \times V_b$ on Γ_{ab} is

$$\sum_{ij} \epsilon_{ab} \epsilon_{ij} a_{ia} b_{jb} - (-1)^m \gamma$$

and on any other cycle Γ_{rs} it is

$$\sum_{ij} \epsilon_{ab} \epsilon_{ij} a_{ir} b_{js}.$$

Hence the period of
$$\sum_{k,a,b} f_{ak} a_{kb} U_a \times V_b$$
on Γ_{rs} is
$$\begin{aligned}
&\sum_{a,b,i,j,k} f_{ak} a_{kb} \epsilon_{ab} \epsilon_{ij} a_{jr} b_{js} - (-1)^m \gamma \sum_k f_{rk} a_{ks} \\
&= \sum_{a,b,j,k} f_{ak} a_{kb} \epsilon_{ab} f_{rj} b_{js} - (-1)^m \gamma \sum_k f_{rk} a_{ks} \\
&= \sum_j f_{rj} a_{js} \left[\sum_{akb} f_{ak} a_{kb} \epsilon_{ab} - (-1)^m \gamma \right], \text{ (since } a = b) \\
&= 0.
\end{aligned}$$

8. Now let us take as our base for cycles of $2m$-dimensions in $A \times B$, $R_m^2 - 1$ of the cycles Γ_{rs}, the cycle Γ_1, and the cycles $a_i^h \times b_{2m-i}^k$ $(i \neq m)$. These last cycles can all be submerged in the manifold of $A \times B$ which corresponds to a section of Φ, and Γ_1 is homologous to the difference between an algebraic cycle and these cycles. Further all these cycles meet Γ_{rs} in a number of points algebraically equal to zero. Hence[6] if an integral of the second kind has zero periods on all the cycles Γ_{rs} it is improper. Therefore

$$\sum_{k,a,b} f_{ak} a_{kb} U_a \times V_b$$

is improper. We write this

$$f^m a^m V U$$

In general if $u_1, \ldots, u_{Rm}$ is any system of integrals of the second kind on A, and $v_1, \ldots, v_{Rm}$ is the corresponding system on B

$$U = \omega^{-1}u, \quad V = \omega^{-1}v$$

where ω is the period matrix of the integrals and we have the result that if α is a matrix

$$\alpha = \bar{\omega}^{-1}f^m a^m \bar{\omega}^{-1}$$

$$\sum_{ij} \alpha_{ij}u_i \times v_j$$

is improper.

9. It is possible to deduce a similar relation between the p- and q-fold integrals of total differentials of the second kind by considering the cycle Γ_r^h of §1. Until, however, some application of such a result arises there is no point in carrying through the analysis, which does not introduce any new idea.

[1] Hodge, *J. Lon. Math. Soc.*, **5**, p. 283 (1930).
[2] Lefschetz, *Colloquium Lectures on Topology*, p. 266 (1930).
[3] Loc. cit.
[4] Hurwitz, *Mat. Ann.*, **28**, 561–585 (1887).
[5] Cf. Baker, *Abel's Theorem and the Allied Theory*, p. 185.
[6] Lefschetz, *Trans. Amer. Math. Soc.*, **22**, 337 (1921).

PROOF OF A RECURRENCE THEOREM FOR STRONGLY TRANSITIVE SYSTEMS

BY GEORGE D. BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated November 27, 1931

Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of n differential equations of the first order, valid in a closed analytic n-dimensional manifold without singularity, M. The points of M are taken to be represented by a finite number of such sets of variables (x) in overlapping domains. For definiteness, the right-hand members X_i as well as the transformations of connection between the sets (x) are taken to be analytic. Finally it will be assumed that there is a volume integral invariant, $\int dx_1\, dx_2 \ldots dx_n$ in suitable coördinates.